Properties of the Graphs of Sinusoids

The general sine wave:

Characteristics of the graph of $y = a\sin\left[b(x-c)\right]+d$ or $y = a\cos\left[b(x-c)\right]+d$:

- Amplitude: $|a|$
- Period: $P = 2\pi/b$
- Frequency: $F = 1/P = b/2\pi$
- Phase shift: c (Remember that the sign of c is the opposite of the sign in the equation).
    - Shift right for $c > 0$.
    - Shift left for $c < 0$.
- Vertical translation: d
    - Shift up for $d > 0$.
    - Shift down for $d < 0$.

Steps to graph $\boldsymbol{y = a\sin\left[b(x-c)\right]+d}$ or $\boldsymbol{y = a\cos\left[b(x-c)\right]+d}$:

Start with the five key points on the graph of $y = \sin x$ or $y = \cos x$.

1. Find five key points for $y = a\sin\left[b(x-c)\right]+d$ or $y = a\cos\left[b(x-c)\right]+d$ by
    a. dividing each x-coordinate by b and adding c.
    b. multiplying each y-coordinate by a and adding d.
    c. sketch one cycle of your graph through the five new points.

***Note:** Order is important. Multiply or divide first, then add.

Examples: Determine the amplitude, period, frequency, phase shift, and vertical shift of the following. Then sketch one cycle of each graph. Draw and label your own axes.

$$y = \sin\left[\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right] + 1$$

$$y = 2\cos\left(2x + \frac{\pi}{2}\right) - 2$$

$$y = -3\cos\left[3\left(x - \frac{2\pi}{3}\right)\right] - 1$$

$$y = 4\sin\left(\frac{\pi}{2}x + \frac{3\pi}{2}\right)$$

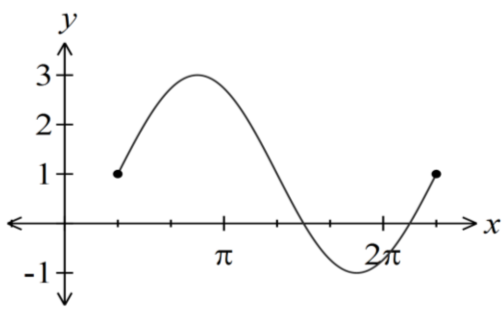
Examples: Find the equation of each sine wave in its final position.

1. The graph of $y = \sin x$ is stretched by a factor of 2, reflected in the *x*-axis, shifted $\pi/5$ units to the right, then translated 4 units downward.

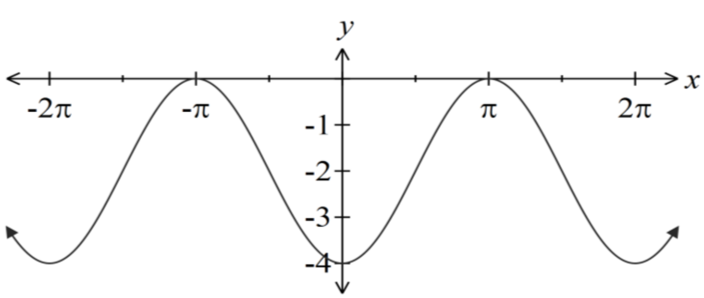
2. The graph of $y = \cos x$ is shifted $\pi/3$ units to the left, translated upward 2 units, then stretched by a factor of 2.

Examples: Find an equation of the requested form whose graph is the given sine wave.

$y = a\sin(x - c) + d$



$y = a\cos(x - c) + d$



Example The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution. The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve: $H(t) = A\cos(\omega t + \phi) + B$, for time *t* and height *H*. A phase shift is not required.

With a diameter of 135 meters, the wheel has a radius of 67.5 meters. The height will oscillate with amplitude 67.5 meters above and below the horizontal center of the wheel.

Passengers board 2 meters above the ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ meters above ground level. The horizontal midline of the oscillation will be at 69.5 meters.

Putting this all together, we have

- Period: $30 = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{\pi}{15}$
- Amplitude: $|A| = 67.5$; $A = -67.5$ (due to the vertical reflection of the cosine curve)
- Phase Shift: $0 = -\frac{\phi}{\omega} \Rightarrow \phi = 0$
- Vertical Shift: $B = 69.5$

An equation for the rider's height, with *t* in minutes and *H* in meters, is

$$H(t) = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$